

Exercise 11

(a) Use the binomial formula (14), Sec. 3, and de Moivre's formula (Sec. 8) to write

$$\cos n\theta + i \sin n\theta = \sum_{k=0}^n \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k \quad (n = 0, 1, 2, \dots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

and use the above summation to show that [compare with Exercise 10(a)]

$$\cos n\theta = \sum_{k=0}^m \binom{n}{2k} (-1)^k \cos^{n-2k} \theta \sin^{2k} \theta \quad (n = 0, 1, 2, \dots).$$

(b) Write $x = \cos \theta$ in the final summation in part (a) to show that it becomes a polynomial*

$$T_n(x) = \sum_{k=0}^m \binom{n}{2k} (-1)^k x^{n-2k} (1-x^2)^k$$

of degree n ($n = 0, 1, 2, \dots$) in the variable x .

Solution**Part (a)**

The binomial theorem states that for two complex numbers, z_1 and z_2 ,

$$(z_1 + z_2)^n = \sum_{k=0}^n \binom{n}{k} z_1^k z_2^{n-k} \quad (n = 1, 2, \dots),$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Set $z_1 = i \sin \theta$ and $z_2 = \cos \theta$.

$$(i \sin \theta + \cos \theta)^n = \sum_{k=0}^n \binom{n}{k} (i \sin \theta)^k (\cos \theta)^{n-k}$$

Apply de Moivre's theorem on the left.

$$\cos n\theta + i \sin n\theta = \sum_{k=0}^n \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k \quad (1)$$

*These are called *Chebyshev polynomials* and are prominent in approximation theory.

Take the complex conjugate of both sides.

$$\cos n\theta - i \sin n\theta = \sum_{k=0}^n \binom{n}{k} \cos^{n-k} \theta (-i \sin \theta)^k \quad (2)$$

Add the respective sides of equations (1) and (2).

$$\begin{aligned} 2 \cos n\theta &= \sum_{k=0}^n \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k + \sum_{k=0}^n \binom{n}{k} \cos^{n-k} \theta (-i \sin \theta)^k \\ &= \sum_{k=0}^n \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k + \sum_{k=0}^n (-1)^k \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k \\ &= \sum_{k=0}^n [1 + (-1)^k] \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k \end{aligned} \quad (3)$$

Notice that if k is odd, the summand is zero. The series can thus be simplified (that is, made to converge faster) by substituting $k = 2p$. Suppose first that n is even.

$$2 \cos n\theta = \sum_{2p=0}^n (2) \binom{n}{2p} \cos^{n-2p} \theta (i \sin \theta)^{2p}$$

Consequently,

$$\cos n\theta = \sum_{p=0}^{\frac{n}{2}} \binom{n}{2p} \cos^{n-2p} \theta (i \sin \theta)^{2p} \quad \text{if } n \text{ is even.} \quad (4)$$

Suppose secondly that n is odd. Then equation (3) can be written as

$$2 \cos n\theta = \sum_{k=0}^{n-1} [1 + (-1)^k] \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k + \underbrace{[1 + (-1)^n] \binom{n}{n} \cos^{n-n} \theta (i \sin \theta)^n}_{=0}$$

Substitute $k = 2p$ as before to simplify the series.

$$2 \cos n\theta = \sum_{2p=0}^{n-1} (2) \binom{n}{2p} \cos^{n-2p} \theta (i \sin \theta)^{2p}$$

Consequently,

$$\cos n\theta = \sum_{p=0}^{\frac{n-1}{2}} \binom{n}{2p} \cos^{n-2p} \theta (i \sin \theta)^{2p} \quad \text{if } n \text{ is odd.} \quad (5)$$

If we define m to be

$$m = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases},$$

then equations (4) and (5) can be combined like so. (p is just a dummy variable and can be replaced with k .)

$$\cos n\theta = \sum_{p=0}^m \binom{n}{2p} (-1)^p \cos^{n-2p} \theta \sin^{2p} \theta$$

Part (b)

The result of part (a) can be written as

$$\begin{aligned}\cos n\theta &= \sum_{p=0}^m \binom{n}{2p} (-1)^p (\cos \theta)^{n-2p} (\sin^2 \theta)^p \\ &= \sum_{p=0}^m \binom{n}{2p} (-1)^p (\cos \theta)^{n-2p} (1 - \cos^2 \theta)^p.\end{aligned}$$

Let $x = \cos \theta$. Then $\theta = \cos^{-1} x$, and the right side becomes a polynomial in x .

$$\cos(n \cos^{-1} x) = \sum_{p=0}^m \binom{n}{2p} (-1)^p x^{n-2p} (1 - x^2)^p$$

$T_n(x) = \cos(n \cos^{-1} x)$ are known as the Chebyshev polynomials.